John Domains, Quasidisks, and the Nehari Class

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1 Introduction

In 1949, Nehari showed in [12] that if f is analytic in the unit disk $\mathbf{D}$ and if its Schwarzian derivative

$$Sf = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$$

satisfies

$$|Sf(z)| \leq \frac{2}{(1-|z|^2)^2}\,, \tag{1.1}$$

then f is univalent in the disk. A necessary condition for univalence, with a 6 replacing the 2 in the nunerator of (1.1), had been proved by Krauss in 1932, [11]. It was rediscovered by Nehari and included in his 1949 paper.

Since Nehari's paper, investigations on the connections between the Schwarzian and univalence have gone primarily in two, often allied, directions. One line of research aimed at establishing other univalence criteria depending on the Schwarzian or related quantities. Another was opened up by an important result of Ahlfors and Weill [1] relating the growth of the Schwarzian to quasiconformal extension.

They proved that if Nehari's condition is strengthened to

$$|Sf(z)| \leq \frac{2t}{(1-|z|^2)^2}, \quad 0 \leq t < 1, \tag{1.2}$$

then, not only is f univalent in the disk, it has a quasiconformal extension to the sphere.

A third area of interest is in general properties of functions satisfying either (1.1) or (1.2), much as one would study special subclasses of univalent functions. Questions of this type are the main concern in the present paper, hence the reference to 'the Nehari Class' in the title. To cite both some older and more recent work as examples, Paatero showed in [16] that convex univalent functions satsify (1.1). This was later proved in a different way by Nehari in [13], who also showed that a bounded convex map satifies (1.2). Gehring and Pommerenke showed in [7] that if f satisfies (1.1) then it has a spherically continuous extension to $\overline{\mathbf{D}}$ and that $f(\mathbf{D})$ is either a Jordan domain or the image of a parallel strip under a Möbius transformation.

Since $S(T \circ f) = Sf$ for any Möbius transformation T many such theorems associated with the Schwarzian are independent of any particular normalization of f. Nevertheless, for some results a normalization is preferable, or required. In this paper, when we refer to a *normalized function* f we mean one with $f(0) = 0$, $f'(0) = 1$, and $f''(0) = 0$. In [2] some sharp distortion theorems and estimates for Hölder continuity associated with the bounds (1.1) and (1.2) were proved assuming this normalization.

There are corresponding normalized extremal functions for (1.1) and (1.2). We fix the notation

$$L(z) = \frac{1}{2}\log\frac{1+z}{1-z} = z + \sum_{n=1}^{\infty}\frac{z^{2n+1}}{2n+1} \tag{1.3}$$

for the normalized logarithm. The function L plays a special role in many of the problems we consider, both because

$$SL(z) = \frac{2}{(1-z^2)^2}\,.$$

and because its image is a parallel strip, in this case the strip $|\mathrm{Im} w| < \pi/4$. The normalized extremal for the Ahlfors-Weill condition (1.2) is

$$A_t(z) = \frac{1}{\alpha}\frac{(1+z)^\alpha - (1-z)^\alpha}{(1+z)^\alpha + (1-z)^\alpha}, \quad \alpha = \sqrt{1-t}. \tag{1.4}$$

We let N denote the set of analytic functions in the disk satisfying (1.1), N^* the elements of N which are not Möbius conjugations of the function L. We use the notation N_0 and N_0^* to indicate the classes of normalized functions.

If $f(z) = z + a_2 z^2 + \ldots$ is in any of the classes, then $f/(1 + a_2 f)$ is in the corresponding class of normalized functions, the point being that the normalized function is still analytic, [2]. In Section 3 we shall also consider meromorphic functions satisfying (1.1). The definition of the Schwarzian is extended to a pole or to the point at ∞ by inversion.

We will discuss definitions and additional background material in Section 2, but for the reader already familiar with the terms in the title, at least, we would like to mention briefly some of our main results. In Section 3 we obtain several characterizations of meromorphic functions satisfying the Nehari condition. We also establish sharp estimates for the modulus of continuity and compactness results for the class N. One of the characteristic properties is in terms of the hyperbolic convexity of powers of the Poincaré metric of the image. This fact appears, either explicitly or implicitly, in much of our later work.

Section 4 is concerned with some global mapping properties of functions in N and N_0. There we concentrate on John domains and quasidisks. We show that if $f \in N$ and if $f(\mathbf{D})$ is a John domain, then $f(\mathbf{D})$ is linearly connected, and hence is a quasidisk, an implication that is certainly false for an arbitrary conformal mapping.

We also show that if $f \in N_0$ and $f(\mathbf{D})$ is not a John domain, then it must already contain an 'infinitesimal strip'. To address these questions we develop several characterizations of conformal mappings onto John domains, both for functions in N_0 and for general conformal mappings of the disk.

Many extremal functions for univalence crtiteria or criteria for quasiconformal extension satisfy $|Sf(z)| \leq Sf(|z|)$. In Section 5 we see how this property leads to versions of some of the results in Section 4 that incorporate the local notion of a well-accessible boundary point.

Finally, in Section 6 we construct examples of two functions in N_0^*, one whose image is linearly connected, but not a John domain, and one whose image is not linearly connected.

2 Background

Here we collect some background material to which we shall refer in the later Sections. We already mentioned the invariance property of the Schwarzian, $S(T \circ f) = Sf$ for Möbius transformations T. We also recall the chain rule for the Schwarzian of the composition of two functions,

$$S(f \circ h) = (S(f) \circ h)(h')^2 + Sh\,.$$

Next, if f is normalized then

$$f(z) = \int_0^z u^{-2}(\zeta)\, d\zeta.$$

where u is the solution of the initial value problem

$$u'' + \frac{1}{2}(Sf)u = 0, \quad u(0) = 1,\ u'(0) = 0. \tag{2.1}$$

The initial value problem (2.1) and standard comparison theorems allow one to deduce a number of inequalities for f, [2]. Thus, if $f \in N_0$ then

$$|f'(z)| \leq L'(|z|) \quad \text{and} \quad |f(z)| \leq L(|z|). \tag{2.2}$$

In either case, equality holds at any $z \neq 0$ if and only if f is a rotation of L. See also the paper of Essèn and Keogh [5].

Also, if $f \in N_0$ then

$$\left|\frac{f''}{f'}(z)\right| \leq \frac{2|z|}{1-|z|^2} = \frac{L''(|z|)}{L'(|z|)}, \tag{2.3}$$

and if equality holds at a single $z \neq 0$ then f must be a rotation of L; this is Lemma 1 in [3]. We shall use this inequality repeatedly. Furthermore, a straightforward adaptation of that argument, which was again a comparison of solutions of two differential equations, can be used to show that if f and F are analytic, locally univalent, and normalized functions on $\mathbf{D}$, with $|Sf(z)| \leq SF(|z|)$, then

$$\left|\frac{f''(z)}{f'(z)}\right| \leq \frac{F''(|z|)}{F'(|z|)}. \tag{2.4}$$

Equality holds for any $z \neq 0$ if and only if $f = F$. This comes up in Section 5. The standard distortion theorem for the full calss of univalent functions in the disk is based on the inequality

$$\left|\frac{1}{2}\frac{f''}{f'}(z) - \frac{\bar{z}}{1-|z|^2}\right| \leq \frac{2|z|}{1-|z|^2}. \tag{2.5}$$

This is often reduced to

$$\left|\frac{f''}{f'}(z)\right| \le \frac{4+2|z|}{1-|z|^2} = \frac{k''(|z|)}{k'(|z|)} \tag{2.6}$$

where $k(z) = z/(1-z)^2$ is the Koebe function.

Finally, if $f \in N_0$ is not Möbius conjugate to L then it is bounded on $\overline{\mathbf{D}}$, [4]. This is very helpful to know, and we use it on several occasions.

Three global geometric notions that are important in our work are that of a John domain, a linearly connected domain, and a quasidisk. Many equivalent definitions for each of these are now known. Briefly, a simply-connected domain Ω is a *John domain* if it is bounded and if there is a constant $a > 0$ such that for every crosscut C of Ω the inequality

$$\operatorname{diam} H \le a \operatorname{diam} C \tag{2.7}$$

holds for one of the components H of $\Omega \backslash C$.

A simply- connected domain Ω is *linearly connected* if there is a constant $b > 0$ such that any two points z_1, z_2 in Ω can be joined by a curve γ with

$$\operatorname{diam} \gamma \le b|z_1 - z_2|. \tag{2.8}$$

A *quasidisk* is a linearly connected, John domain. This is not the standard definition, *i.e.*, the image of a disk under a quasiconformal mapping of $\overline{\mathbf{C}}$, since here we are requiring *a priori* that the domain be bounded. However, the definition we adopt involves no real loss of generality and is better suited to our needs. Intuitively, the John condition prohibits outward pointing cusps and the linearly connected condition prohibits inward pointing cusps. We refer to [18] for all of these definitions, and also to [6] for this definition of a John domain (which is also not the original definition).

It is actually an equivalent, analytical characterization of John domains that will be the basis of our work in Section 4.

We also need the idea of a well-accessible boundary point, see [18], Chapter 11. This is a local condition. Let f be a conformal mapping of $\mathbf{D}$ onto a domain Ω and suppose that f has an angular limit at a point ζ, $|\zeta| = 1$. Then $f(\zeta)$ is *well-accessible* if there is a Jordan arc γ in $\mathbf{D}$ ending at ζ and a constant $M > 0$, such that

$$\operatorname{diam} f(\gamma(z)) \le M d(f(z), \partial\Omega), \tag{2.9}$$

where $\gamma(z)$ denotes the part of γ from z to ζ. If f is a conformal mapping onto a John domain then all the bounary points $f(\zeta)$ are well-accessible, with a constant M independent of ζ. Conversely, if all boundary points are uniformly well-accessible, then Ω is a John domain

Recall that the Poincaré metric $\lambda_\Omega|dw|$ of a simply connected domain Ω is defined by

$$\lambda_\Omega(f(z))|f'(z)| = \lambda_{\mathbf{D}}(z) = \frac{1}{1-|z|^2}, \tag{2.10}$$

where $f\colon D \to \Omega$ is a conformal mapping of the unit disk onto Ω. We let $h_{\mathbf{D}}$ and h_Ω denote the hyperbolic distance in $\mathbf{D}$ and in Ω, respectively. Hyperbolic distance is invariant under conformal mapping.

From Schwarz's lemma and the Koebe 1/4−theorem one has the sharp inequalities

$$\frac{1}{4}\frac{1}{d(z,\partial\Omega)} \le \lambda_\Omega(z) \le \frac{1}{d(z,\partial\Omega)}, \tag{2.11}$$

where $d(z,\partial\Omega)$ denotes the Euclidean distance from z to the boundary. Equivalently

$$d(z,\partial\Omega) \le (1-|z|^2)|f'(z)| \le 4d(z,\partial\Omega). \tag{2.12}$$

In Section 3 we will be concerned with critical points and minima for λ_Ω. If $f \in N$ is bounded then λ_Ω has a unique critical point, [3].

Also, quite generally, a real-valued function v on Ω is *hyperbolically convex* if $(v\circ\gamma)''(t) \ge 0$ for each hyperbolic geodesic $\gamma = \gamma(t)$ in Ω. The hyperbolic convexity can also be defined in terms of the positive definiteness, or semi-definiteness, of the Hessian of v, computed with respect to the Poincaré metric. It is an invariant notion.

These definitions can be extended by use of the local coordinate $1/z$ to include the case when ∞ is an interior point of Ω, so when f is a meromorphic conformal mapping of the disk onto Ω. If $\infty \in \Omega$ then $\lambda_\Omega = O(|z|^{-2})$ as $z \to \infty$. For hyperbolic convexity we could also simply declare v to be hyperbolically convex in Ω if it is so in $\Omega\backslash\{\infty\}$.

Finally, recall that if f is analytic at $z_0 \in \mathbf{D}$ with $f'(z_0) \ne 0$ then the Koebe transform of f is defined by

$$\frac{f\left(\dfrac{z+z_0}{1+\bar{z}_0 z}\right) - f(z_0)}{(1-|z_0|^2)f'(z_0)} = z + A_2(z_0)z^2 + A_3(z_0)z^3 + \cdots,$$

where

$$A_2(z_0) = \frac{1}{2}(1-|z_0|^2)\frac{f''}{f'}(z_0) - \bar{z}_0. \tag{2.13}$$

3 Characterizations, Continuity, and Compactness

For our work in this Section we opt for slightly greater generality and consider meromorphic functions satisfying the Nehari condition (1.1). There are several reasons for doing this. The primary one is that we are able to find characterizations of such functions in terms of the Poincaré metric of the image if we allow for shifting the image by Möbius transformations. Since this could introduce a pole of the mapping it is more natural to consider meromorphic functions at the outset, even though we can often reduce to the analytic case. The result is as follows:

Theorem 1 *Let f be meromorphic and univalent in $\mathbf{D}$ and let $\Omega = f(\mathbf{D})$. The following are equivalent.*

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(i) f satisfies (1.1).

(ii) If T is any Möbius transformation, then the function $\lambda^\alpha_{T(\Omega)}$ is hyperbolically convex for every $\alpha \ge 1/2$.

(ii)′ If T is any Möbius transformation, then the function $\lambda^{\alpha}_{T(\Omega)}$ is hyperbolically convex for some $\alpha \geq 1/2$.

(iii) For each $z_0 \in \mathbf{D}$ there is a Möbius transformation T such that $\infty \notin T(\Omega)$ and $\lambda_{T(\Omega)}$ has a global minimum at $T(f(z_0))$.

(iii)′ For each $z_0 \in \mathbf{D}$ there is a Möbius transformation T such that $\infty \notin T(\Omega)$ and $\lambda_{T(\Omega)}$ has a local minimum at $T(f(z_0))$.

A paper of Yamashita [21] contains some statements of a related nature, in a different context. The proof of $(iii)' \Rightarrow (i)$ will require a result which we prefer to state as a general Lemma on analytic functions rather than as a property of the metric. This was also observed in [21]. We give a simple proof here for completeness.

Lemma 1 *Let $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ be analytic in a neighborhood of $z = 0$, and suppose that $(1 - |z|^2)|f'(z)|$ has a local maximum at $z = 0$. Then $a_2 = 0$, $|a_3| \leq (1/3)|a_1|$, and hence $|Sf(0)| \leq 2$.*

It is not difficult to give examples to show that the converse is not true.

Proof. We may assume that $a_1 = 1$. Then $(1 - |z|^2)|f'(z)| \leq 1$ near 0. Expanding into a series gives

$$\begin{aligned}&(1 - |z|^2)|f'(z)| = \\ &\quad (1 - |z|^2)(1 + 2\mathrm{Re}\{a_2 z\} + 3\mathrm{Re}\{a_3 z^3\} + 2(\mathrm{Im}\{a_3 z\})^2 + O(z^3)).\end{aligned} \tag{3.1}$$

Hence

$$1 + 2\mathrm{Re}\{a_2 z\} + O(z^2) \leq 1,$$

as $|z| \to 0$, which implies that $a_2 = 0$. Using this in (3.1) gives that

$$1 + 3\mathrm{Re}\{a_3 z^2\} - |z|^2 + O(z^3) \leq 1,$$

as $|z| \to 0$, and we conclude that $|a_3| \leq 1/3$.

Proof of Theoorem 1. $(i) \Rightarrow (ii)$: Write $g = T \circ f$ and

$$A_2(z) = \frac{1}{2}(1 - |z|^2)\frac{g''}{g'}(z) - \bar{z}, \tag{3.2}$$

see (2.13).

The function $\lambda^{\alpha}_{T(\Omega)}$ will be hyperbolically convex in $T(\Omega)\backslash\{\infty\}$ if and only if $[(1 - |z|^2)|g'(z)|]^{-\alpha}$ is hyperbolically convex in $\mathbf{D}$, away from a pole of g.

This translates to the inequality

$$\mathrm{Re}\{\zeta^2(1 - |z|^2)^2 Sg(z)\} - \left(\alpha - \frac{1}{2}\right)\mathrm{Re}\{\zeta A_2(z)\}^2 - 2[\mathrm{Im}\{\zeta A_2(z)\}]^2 \leq 2 \tag{3.3}$$

for all $|\zeta| = 1$. The points ζ on the unit circle parametrize the directions of the hyperbolic geodesics.

We will not give the details of the derivation of (3.3), but it amounts to this. To make the calculation easier we can suppose that $z = 0$ is not a pole of g; once we have the general condition then this is irrelevant. First, compute the second x-derivative of $[(1-|z|^2)|g'(z)|]^{-\alpha}$ at the origin. This produces, fairly easily, the inequality at $z = 0$ with $\zeta = 1$. ¿From this, the invariance of $[(1-|z|^2)|g'(z)|]^{-\alpha}$ under Möbius transformations of **D** onto itself leads to the general inequality.

From the fact that $S(T \circ f) = Sf$ for any Möbius transformation T, it is clear that if f satisfies the Nehari condition (1.1) then (3.3) holds for all $\alpha \geq 1/2$. Therefore $\lambda^{\alpha}_{T(\Omega)}$ is hyperbolically convex away from ∞ if $\infty \in T(\Omega)$, and hence it is hyperbolically convex everywhere in $T(\Omega)$.

The implication $(ii) \Rightarrow (ii)'$ is trivial. Suppose now that $(ii)'$ holds. In general the quantity $(1-|z|^2)|g'(z)|$, g analytic in **D**, $g' \neq 0$, will have a critical point at z_0 if and only if $A_2(z_0) = 0$, referring to (3.2).

With $g = T \circ f$ and

$$\frac{g''}{g'}(z) = \frac{T''}{T'}(f(z))f'(z) + \frac{f''}{f'}(z),$$

it is easy to see that there are enough parameters available to find a Möbius transformation T such that $\infty \notin T(\Omega) = g(\mathbf{D})$, *i.e.*, g is analytic in **D**, and $A_2(z_0) = 0$. The hyperbolic convexity of $\lambda^{\alpha}_{T(\Omega)}$ for any single α implies that z_0 must give a global maximum of $(1-|z|^2)|g'(z)| = \lambda_{T(\Omega)}(g(z))^{-1}$, and so a global minimum of $\lambda_{T(\Omega)}$.

$(iii) \Rightarrow (iii)'$ is again trivial. Suppose that $(iii)'$ holds. Let $z_0 \in \mathbf{D}$ and let T be a Möbius transformation such that $g = T \circ f$ is analytic and $(1-|z|^2)|g'(z)|$ has a local maximum at $z = z_0$. Define

$$h(w) = g\left(\frac{w + z_0}{1 + \bar{z}_0 w}\right).$$

Then h is analytic at $w = 0$ and $(1-|w|^2)|h'(w)|$ has a local maximum there. Lemma 1 implies that

$$(1-|z_0|^2)|Sf(z_0)| = |Sh(0)| \leq 2.$$

This establishes the implication $(iii)' \Rightarrow (i)$ and completes the proof of the Theorem.

The condition (3.3) resembles

$$(1-|z|^2)^2|Sf(z)| + 2\left|\bar{z} - \frac{1}{2}(1-|z|^2)\frac{f''}{f'}(z)\right|^2 \leq 2, \tag{3.4}$$

which appears in a paper of Kim and Minda [10]. There it is shown that (3.4) holds if and only if f is a convex conformal mapping, if and only if $1/\lambda_\Omega$ is concave, if and only if $\log \lambda_\Omega$ is convex. All the statements here on convexity are with respect to the *Euclidean* metric. Recall that the class of convex conformal mappings is contained in N.

We do not know what might happen with the hyperbolic convexity of $\lambda^{\alpha}_{\Omega}$ when $0 < \alpha < 1/2$. Also, it is worth pointing out a special case of the Theorem for the class N_0 of analytic, normalized functions satisfying (1.1).

Corollary 1 *If $f \in N_0$, $\Omega = f(\mathbf{D})$, then $1 = \lambda_\Omega(0) \leq \lambda_\Omega(w)$ for all $w \in \Omega$. If equality holds at any $w \neq 0$ then f is a rotation of L.*

Proof. Taking the $\partial_z = \partial/\partial z$ derivative of the logarithm of (2.10) gives

$$\frac{\partial_w \lambda_\Omega(w)}{\lambda_\Omega(w)} f'(z) = \frac{\bar{z}}{1-|z|^2} - \frac{1}{2}\frac{f''}{f'}(z)\,, \quad w = f(z). \tag{3.5}$$

Thus for $f \in N_0$, λ_Ω has a critical point at $w = 0$, and hence, by convexity, the global minimum is $\lambda_\Omega(0) = 1$. By (2.3), if there is any other critical point then f must be a rotation of L. If this is so then its image Ω is a parallel strip, and λ_Ω takes its minimum value 1 all along the center line.

Interestingly, Theorem 1 also leads to a characterization of functions in N^* in terms of the Ahlfors-Weill quasiconformal extension of functions which satisfy (1.2). Let

$$\begin{aligned} E_f(\zeta) &= f(\zeta) + \frac{(1-|\zeta|^2)f'(\zeta)}{\bar{\zeta} - \dfrac{1}{2}(1-|\zeta|^2)\dfrac{f''}{f'}(\zeta)} \\ &= f(z) + \frac{1}{\partial_w(\log \lambda_\Omega)(f(z))}. \end{aligned} \tag{3.6}$$

If T is a Möbius transformation, then

$$E_{T\circ f} = T(E_f). \tag{3.7}$$

If f satisfies (1.2) then

$$F(z) = \begin{cases} f(z) & |z| \leq 1 \\ E_f(1/\bar{z}) & |z| > 1 \end{cases} \tag{3.8}$$

is a $\frac{1+t}{1-t}$-quasiconformal mapping which extends f, and in [3] it was shown that if $f \in N^*$ then already F defines a homeomorphic extension of f to $\overline{\mathbf{C}}$.

Corollary 2 *Let f be univalent in $\mathbf{D}$ and let $\Omega = f(\mathbf{D})$. The following are equivalent.*

(i) $f \in N^*$.

(ii) E_f is injective with values in $\overline{\mathbf{C}} \backslash \overline{f(\mathbf{D})}$.

(iii) For each $z_0 \in \mathbf{D}$ there is a Möbius transformation T such that $\infty \notin \overline{T(\Omega)}$ and $\lambda_{T(\Omega)}$ has a unique global minimum at $T(f(z_0))$.

Proof. The implication $(i) \Rightarrow (ii)$ is clear by the remarks on homeomorphic extensions, above. Suppose that (ii) holds. Fix $z_0 \in \mathbf{D}$ and choose a Möbius transformation T such that $(T \circ E_f)(z_0) = \infty$. Using (3.7), we first have that $E_{T\circ f}(z_0) = \infty$, and then because $E_{T\circ f}$ takes values in $\overline{\mathbf{C}} \backslash \overline{T(\Omega)}$, that $\infty \notin \overline{T(\Omega)}$, *i.e.*, that $T(\Omega)$ is bounded. Therefore $\lambda_{T(\Omega)}(w) \to \infty$ as $w \to \partial T(\Omega)$, by (2.11), so there is at least one critical point of $\lambda_{T(\Omega)}$ in $T(\Omega)$. One such point is $T(f(z_0))$, because $E_{T\circ f}(z_0) = \infty$, and by (3.6) this can only happen if $\lambda_{T(\Omega)}$ has a critical point at $T(f(z_0))$. If $T(f(z_1))$ is another critical point then $E_{T\circ f}(z_1) = \infty$ as well. But by (3.7) again, $E_{T\circ f}$ is also injective, and hence $z_1 = z_0$ from the univalence of $T \circ f$. Therefore $\lambda_{T(\Omega)}$ has a unique critical point, which must be a global minimum. This proves $(ii) \Rightarrow (iii)$.

Finally, if (iii) holds then $f \in N$ by Theorem 1. f cannot be Möbius conjugate to L for then its image would be a Möbius transformation of a parallel strip and λ_Ω would have an entire line, or circle, of minima. Thus $f \in N^*$, and the Corollary is proved.

For N_0 we have the basic distortion and growth results (2.3). There is a corresponding Theorem for meromorphic functions, with $1/L$ as the extremal. See also two papers of Steinmetz, [19], [20].

Theorem 2 *Let* $f(z) = \dfrac{1}{z} + a_0 + a_1 z + \cdots$ *satisfy (1.1) Then:*

(i) $\quad |f'(z)| \le \dfrac{1}{(1-|z|^2)L(|z|)^2}$. *If equality holds at any* $z \neq 0$, *then* f *is a rotation of* $1/L$.

(ii) $\quad |f(r_2\zeta) - f(r_1\zeta)| \le \dfrac{1}{L(r_1)} - \dfrac{1}{L(r_2)}$, $0 < r_1 < r_2 < 1$, $|\zeta| = 1$. *If equality holds for any pair* $r_1 < r_2$, *then* f *is a rotation of* $1/L$.

(iii) $\quad |f(z_1) - f(z_2)| \le K\left[\log \dfrac{4}{|z_1 - z_2|}\right]^{-1}$, $0 < r_0 \le |z_1|, |z_2| < 1$. *Here* K *and* r_0 *are absolute constants.*

The function $1/L$ shows that the order of magnitude in (iii) is best possible.

Proof. Let $|\zeta| = 1$ and for $0 < r < 1$ consider

$$\eta_\zeta(r) = [(1-r^2)|f'(r\zeta)|]^{-1/2} = r + O(r^2). \tag{3.9}$$

One finds that

$$\frac{d}{dr}\{(1-r^2)\eta_\zeta'(r)\} = (1-r^2)\eta_\zeta(r)\left\{\frac{1}{(1-r^2)^2} - \frac{1}{2}\mathrm{Re}\{\zeta^2 Sf(r\zeta)\} + \frac{1}{4}\left[\mathrm{Im}\left\{\zeta\frac{f''}{f'}(r\zeta)\right\}\right]^2\right\}. \tag{3.10}$$

This is non-negative by virtue of (1.1). Hence $(1-r^2)\eta_\zeta'(r) \ge \eta_\zeta'(0) = 1$, and

$$\begin{aligned} \eta_\zeta'(r) &\ge \frac{1}{1-r^2} = L'(r), \\ \eta_\zeta(r) &\ge L(r). \end{aligned} \tag{3.11}$$

Then

$$(1-r^2)|f'(r\zeta)| = \frac{1}{\eta_\zeta(r)^2} \le \frac{1}{L(r)^2},$$

which proves the first part of (i).

In order to have equality at a point $z \neq 0$, one would have to have equality in (3.11) for all points on the segment $[0, z]$. This would mean that the derivative in (3.10) is zero there. Hence the real part of the Schwarzian would have to equal to $2/(1-r^2)^2$ while the imaginary part would have to vanish. Therefore the full Schwarzian Sf would be extremal along that segment and we conclude that f is a rotation of $1/L$

The inequality in (ii) follows from (i) and (3.11) by integration:

$$|f(r_2\zeta) - f(r_1\zeta)| \leq \int_{r_1}^{r_2} |f'(r\zeta)|\, dr \leq \int_{r_1}^{r_2} \frac{L'(r)}{L(r)^2}\, dr = \frac{1}{L(r_1)} - \frac{1}{L(r_2)} .$$

The statement on the case of equality follows essentially as above.

For the proof of (iii), we first recall that the hyperbolic geodesic γ between two points z_1, z_2 in the disk has the property that its Euclidean length l is $\leq (\pi/2)|z_1 - z_2|$, and that $\min\{s,\, l-s\} \leq (\pi/2)(1-|z|)$ for each $z \in \gamma$, where s is the Euclidean length of the part of γ between z_1 and z.

Now let A be the annulus $1 - e^{-2} \leq |z| < 1$ and let A' be the annulus $1 - (1/2)e^{-2} \leq |z| < 1$. Choose $0 < \delta < 1$ such that the hyperbolic geodesic γ between two points $z_1, z_2 \in A'$ is contained in A whenever $|z_1 - z_2| \leq \delta$.

In this situation, the geometric property of γ mentioned above, together with (i), and the fact that the function $(1/x)(\log(1/x))^{-2}$ is decreasing for $0 < x \leq e^{-2}$, allow us to deduce that

$$\begin{aligned} |f(z_1) - f(z_2)| &\leq \int_\gamma \frac{4}{(1-|z|^2)} \left[\log \frac{1+|z|}{1-|z|}\right]^{-2} |dz| \\ &\leq 4 \int_\gamma \frac{1}{1-|z|} \left[\log \frac{1}{1-|z|}\right]^{-2} |dz| \\ &\leq 8 \int_0^{l/2} \frac{\pi}{2s} \left[\log \frac{\pi}{2s}\right]^{-2} ds \\ &\leq 4\pi \left[\log \frac{2}{|z_1 - z_2|}\right]^{-1} . \end{aligned}$$

To complete the proof of (iii) when z_1 and z_2 are not close, and then to get a finite upper bound if $|z_1 - z_2|$ is near 2, we first replace the final line above by the weaker estimate

$$|f(z_1) - f(z_2)| \leq 4\pi \left[\log \frac{2}{|z_1 - z_2|}\right]^{-1} \leq 8\pi \left[\log \frac{4}{|z_1 - z_2|}\right]^{-1} ,$$

which holds as soon as $|z_1 - z_2| \leq 1$.

Suppose now that $z_1, z_2 \in A'$ and $|z_1 - z_2| > \delta$. Then there are points $w_1, \ldots, w_{n-1}$ in A' such that $|z_1 - w_1|, |w_1 - w_2|, \ldots, |w_{n-1} - z_2| \leq \delta < 1$ with $n \leq n_0$, n_0 an absolute constant. By the triangle inequality,

$$\begin{aligned} |f(z_1) - f(z_2)| &\leq 8\pi \left\{ \left[\log \frac{4}{|z_1 - w_1|}\right]^{-1} + \cdots + \left[\log \frac{4}{|w_{n-1} - z_2|}\right]^{-1} \right\} \\ &\leq 8n\pi \left[\log \frac{4}{\delta}\right]^{-1} \leq 8n_0\pi \left[\log \frac{4}{|z_1 - z_2|}\right]^{-1} . \end{aligned}$$

This proves (iii) with $r_0 = 1 - (1/2)e^{-2}$ and $K = 8n_0\pi$, and completes the proof of the Theorem.

The hyperbolic convexity of $\lambda_\Omega^{1/2}$ is at work here in the use of (3.9) and (3.10) to prove the basic estimate in part (i), though we have not used it explicitly. One can conclude from (3.3) that the expression in (3.10) is non-negative; it is precisely the convexity of $\lambda_\Omega^{1/2}$ at the point $r\zeta$, $|\zeta| = 1$, along the radial geodesic toward ζ.

There are two quick corollaries for the class N, one on the modulus of continuity and one on compactness.

Corollary 3 *Let $f \in N$ and suppose that $|f(z)| \le M$ for $z \in \mathbf{D}$. Then*

$$|f(z_1) - f(z_2)| \le K_1 M^2 \left[\log \frac{4}{|z_1 - z_2|}\right]^{-1},$$

for all z_1, $z_2 \in \mathbf{D}$, where K_1 is an absolute constant.

Proof. This is a direct consequence of part (iii) of Theorem 2 applied to $1/f$; the cases $|z_1|$ or $|z_2| \le r_0$ are obvious.

Corollary 4 *Let $\{f_n\}$ be a sequence in N and let $M_n = \sup_{\mathbf{D}} |f_n|$.*

(i) If $M_n \le M < \infty$ for all n then there is a subsequence $\{f_{n_k}\}$ which converges uniformly on $\overline{\mathbf{D}}$ to a bounded function in N.

(ii) If $f_n \in N_0$ for all n and if $M_n \to \infty$ then there is a subsequence $\{f_{n_k}\}$ which converges to a rotation F of L, locally uniformly in $\mathbf{D}$. The subsequence can be chosen so that $1/f_{n_k} \to 1/F$ uniformly on $\epsilon \le |z| \le 1$ for every $\epsilon > 0$.

Proof. Part (i) is a direct consequence of Corollary 3 and the Arzela-Ascoli Thoerem. For part (ii), by virtue of the inequality $|f(z)| \le L(|z|)$, from (2.2), we can extract a subsequence $\{f_{n_k}\}$ converging locally uniformly in $\mathbf{D}$ to a map in N_0. Let F be the limit. Since $M_{n_k} \to \infty$, F cannot be bounded, and hence must be a rotation of L. Finally, using Theorem 2, a further subsequence, labelled the same way, will have $1/f_{n_k} \to 1/F$ uniformly on $r_0 \le |z| \le 1$, and the same subsequence will converge uniformly to $1/F$ on every $\epsilon \le |z| \le 1$.

4 John Domains and Quasidisks

We recall the definitions of John domains, linearly connected domains, quasidisks and well-accessible boundary points from Section 2. The central result of this Section is the phenomenon that, within N, if the image of the disk is a John domain then it is a quasidisk. This is Theorem 4, below. It is therefore of additional interest to determine when the image will be a John domain, and we address this question first.

We shall need some characterizations of John domains in terms of a general conformal mapping before bringing in the class N. Theorem 5.2 in [18] furnishes several equivalent statements. We state only part of that Theorem here, reformulated slightly from [18].

For $z \in \mathbf{D}$ define the annular sector

$$B(z) = \{\xi \in \mathbf{D}\colon |z| \leq |\xi| \leq 1,\ |\arg \xi - \arg z| \leq \pi(1-|z|)\}. \tag{4.1}$$

Let f map $\mathbf{D}$ conformally onto a domain Ω. Then Ω is a John domain if and only if there are constants $0 < \alpha \leq 1$ and $0 < M < \infty$ such that

$$\frac{(1-|w|^2)|f'(w)|}{(1-|z|^2)|f'(z)|} \leq M\left(\frac{1-|w|}{1-|z|}\right)^{\alpha}, \quad \text{for } w \in B(z), \tag{4.2}$$

where the estimate holds uniformly in z and w.

We derive from this several other characterizations that are tailored more to our particular applications, and which may be of independent interest. For the first of these see also [17].

Lemma 2 *Let f be analytic and univalent in $\mathbf{D}$. Then $f(\mathbf{D})$ is a John domain if and only if there exists $x < 1$ such that*

$$\sup_{|\zeta|=1} \sup_{r<1} \frac{(1-\rho^2)|f'(\rho\zeta)|}{(1-r^2)|f'(r\zeta)|} < 1, \quad \rho = \frac{x+r}{1+xr}\,. \tag{4.3}$$

Proof. Suppose that $f(\mathbf{D})$ is a John domain. Then according to (4.2) there exist constants $M > 0$ and $0 < \alpha \leq 1$, independent of r, ρ, and ζ, such that

$$\frac{(1-\rho^2)|f'(\rho\zeta)|}{(1-r^2)|f'(r\zeta)|} \leq M\left(\frac{1-\rho}{1-r}\right)^{\alpha} = M\left(\frac{1-x}{1+rx}\right)^{\alpha} \leq M(1-x)^{\alpha}.$$

We can make the last term < 1 by taking x sufficiently close to 1.

Conversely, suppose that

$$\frac{(1-\rho^2)|f'(\rho\zeta)|}{(1-r^2)|f'(r\zeta)|} \leq \beta < 1 \quad \text{for} \quad \rho = \frac{x+r}{1+xr}\,,\ |\zeta| = 1, \tag{4.4}$$

uniformly in r and ζ. We want to show that (4.2) holds.

Let x_k, $k = 1, 2, \ldots$ be defined by

$$\frac{1+x_k}{1-x_k} = \left(\frac{1+x}{1-x}\right)^k,\ \textit{i.e.},\ x_{k+1} = \frac{x+x_k}{1+xx_k}\,.$$

Then (4.4) implies that

$$\frac{(1-x_{k+1}^2)|f'(x_{k+1}|}{(1-x_k^2)|f'(x_k)|} < \beta < 1.$$

Now let $0 < \alpha \leq 1$ be such that

$$\beta < \left(\frac{1+x}{1-x}\right)^{\alpha}.$$

Then for $j < k$

$$\begin{aligned}\frac{(1-x_k^2)|f'(x_k|}{(1-x_j^2)|f'(x_j)|} &< \beta^{k-j} < \left(\frac{1-x_k}{1+x_k}\right)^{\alpha}\left(\frac{1-x_j}{1+x_j}\right)^{-\alpha} \\ &\leq 2^{\alpha}\left(\frac{1-x_k}{1-x_j}\right)^{\alpha}.\end{aligned} \tag{4.5}$$

We claim that (4.5) implies that

$$\frac{(1-s^2)|f'(s\zeta)|}{(1-r^2)|f'(r\zeta)|} \leq M_1\left(\frac{1-s}{1-r}\right)^{\alpha}, \tag{4.6}$$

for all $0 \leq r \leq s < 1$. To see this, let

$$\nu(te^{i\theta}) = \log((1-t^2)|f'(te^{i\theta})|).$$

Then

$$\frac{\partial}{\partial t}\nu(te^{i\theta}) = -\frac{2t}{1-t^2} + \mathrm{Re}\left\{e^{i\theta}\frac{f''}{f'}(t\zeta)\right\},$$

hence $|\partial\nu/\partial t| \leq 8/(1-t^2)$ by the distortion theorem for univalent functions, (2.6). Integrating from r to s gives

$$\frac{(1-s^2)|f'(s\zeta)|}{(1-r^2)|f'(r\zeta)|} \leq e^{8h_{\mathbf{D}}(r,s)}. \tag{4.7}$$

Since $h_{\mathbf{D}}(x_k, x_{k+1}) = h_{\mathbf{D}}(0, x)$ for all k, we obtain (4.6) from (4.5) and (4.7) with a constant M_1 which is independent of ζ.

The inequality (4.6) is condition (4.2) along a radius. To get

$$\frac{(1-|w|^2)|f'(w)|}{(1-|z|^2)|f'(z)|} \leq M\left(\frac{1-|w|}{1-|z|}\right)^{\alpha}, \tag{4.8}$$

uniformly in z and w, when w in the annular sector $B(z)$ but not on the same radius as z, we argue as follows. Let ξ be on the same radius as w with $|\xi| = |z|$. Then

$$\frac{(1-|w|^2)|f'(w)|}{(1-|\xi|^2)|f'(\xi)|} \leq M_1\left(\frac{1-|w|}{1-|\xi|}\right)^{\alpha} = M_1\left(\frac{1-|w|}{1-|z|}\right)^{\alpha}. \tag{4.9}$$

Now along the circular arc from z to ξ,

$$\frac{\partial}{\partial\theta}\nu(te^{i\theta}) = -\mathrm{Im}\left\{te^{i\theta}\frac{f''}{f'}(te^{i\theta})\right\} \leq \frac{6}{1-t^2},\ t = |\xi| = |z|,$$

where we have again used (2.6). Therefore

$$\left|\log\frac{(1-|\xi|^2)|f'(\xi)|}{(1-|z|^2)|f'(z)|}\right| \leq 6\pi,$$

by integration and by the definition of $B(z)$ in (4.1). Then (4.8) follows from this and (4.9). Hence Ω is a John domain, and the proof is complete.

Observe that in the course of the proof we have established that (4.6) implies (4.2). The latter is the characteristic property of a conformal mapping onto a John domain that we stated at the beginning of this Section. The reverse implication is trivial, and we can state:

Corollary 5 *Let f be analytic and univalent in $\mathbf{D}$. Then $f(\mathbf{D})$ is a John domain if and only if there are constants M, $0 < \alpha \leq 1$, such that*

$$\sup_{|\zeta|=1} \frac{(1-s^2)|f'(s\zeta)|}{(1-r^2)|f'(r\zeta)|} \leq M\left(\frac{1-s}{1-r}\right)^{\alpha}, \quad 0 \leq r \leq s < 1, \tag{4.10}$$

This is related to the notion of a well-accessible boundary point, defined in Section 2, (2.9), and leads to still another analytic characterization of John domains. We explain the connection briefly for the purposes of our specific applications, and again we refer to [18] for a more thorough discussion. According to Theorem 11.3 in [18], the condition

$$|f'(s\zeta)| \leq M|f'(r\zeta)|\left(\frac{1-s}{1-r}\right)^{\alpha-1}, \tag{4.11}$$

for $0 \leq r \leq s < 1$, is necessary and sufficient for the boundary point $f(\zeta)$ to be well-accessible. Here the constant M may depend on ζ, and $\alpha > 0$. To be in accord with (4.10) we rewrite this as

$$\frac{(1-s^2)|f'(s\zeta)|}{(1-r^2)|f'(r\zeta)|} \leq M\left(\frac{1-s}{1-r}\right)^{\alpha}, \quad 0 \leq r \leq s < 1. \tag{4.12}$$

We will return to this in Section 5.

Corollary 6 *Let f be analytic and univalent in $\mathbf{D}$. Then $f(\mathbf{D})$ is a John domain if and only if*

$$\sup_{|\zeta|=1} \sup_{0\leq r\leq s<1} \frac{|f(s\zeta)-f(r\zeta)|}{(1-r^2)|f'(r\zeta)|} < \infty. \tag{4.13}$$

Proof. Suppose $f(\mathbf{D})$ is a John domain. Then (4.10) holds by Corollary 5. Now, for a John domain all boundary points are well-accessible, and hence we obtain (4.12). But (4.12) clearly implies

$$\frac{|f(s\zeta)-f(r\zeta)|}{(1-r^2)|f'(r\zeta)|} \leq K, \tag{4.14}$$

by integration, where K depends on M and α, and this proves (4.13).

Next suppose that (4.14) holds, uniformly in ζ. Then

$$\operatorname{diam} f([r\zeta,\zeta]) \leq K(1-r^2)|f'(r\zeta)| \leq 4Kd(f(r\zeta), \partial f(\mathbf{D})),$$

by (2.12). Hence $f(\zeta)$ is well-accessible by definition, (2.9), and so (4.11), whence (4.12) is verified, again by appeal to Theorem 11.3 in [18]. Thus $f(\mathbf{D})$ is a John domain by Corollary 5.

We have a few more comments on these results. In principle, Lemma 2 has nothing to do with bounds on the Schwarzian, but actually it fits in quite well with N_0.

Suppose $f \in N_0$ and $\Omega = f(\mathbf{D})$. We turn again to the function

$$\eta_\zeta(r) = [(1-r^2)|f'(r\zeta)|]^{-1/2} = \lambda_\Omega^{1/2}(f(r\zeta)) = 1 + 0(r^2), \tag{4.15}$$

$0 \le r < 1$, $|\zeta| = 1$. One checks that

$$\frac{\eta_\zeta'}{\eta_\zeta}(r) = \frac{r}{1-r^2} - \frac{1}{2}\mathrm{Re}\left\{\zeta\frac{f''}{f'}(r\zeta)\right\}, \tag{4.16}$$

The expression on the right of (4.16) is positive by (2.3), and strictly positive if f is not a rotation of L. Hence η_ζ is an increasing function of r, and

$$\frac{(1-\rho^2)|f'(\rho\zeta)|}{(1-r^2)|f'(r\zeta)|} \le 1 \quad \text{for} \quad r \le \rho. \tag{4.17}$$

The inequality is strict except when f is a rotation of L. Thus the stronger inequality

$$\frac{(1-\rho^2)|f'(\rho\zeta)|}{(1-r^2)|f'(r\zeta)|} \le k < 1$$

of Lemma 2 is a reasonable condition to impose on a function $f \in N_0$.

Both (4.17) and Lemma 2 have a natural, and similar, formulation in terms of λ_Ω. Each hyperbolic geodesic starting at $0 = f(0)$ in Ω is the image of a radius $f(r\zeta)$ for $0 \le r < 1$, with the directions parametrized by ζ. Then (4.17) says that $\lambda_\Omega(w_1) \le \lambda_\Omega(w_2)$ if w_1 and w_2 lie in that order on a hyperbolic geodesic starting at $w_0 = 0$. Lemma 2 says that a necessary and sufficient condition for a bounded, simply connected domain Ω to be a John doimain is that there exist constants, $\delta > 0$, $k < 1$, and a point $w_0 \in \Omega$ such that $\lambda_\Omega(w_1) \le k\lambda_\Omega(w_2)$ if w_1 and w_2 lie in that order on a hyperbolic geodesic starting from w_0, and $h_\Omega(w_1, w_2) = \delta\,(= h_{\mathbf{D}}(0, x))$. The constant k is independent of the geodesic from w_0.

Despite the similarity in the statements, one way of pointing out the difference is this. In the first case, if $f \in N_0$, then $w_0 = 0$ is a minimum point for λ_Ω (and $\lambda_\Omega(0) = 1$). In the case of Lemma 2, though one can normalize to get $w_0 = 0$ and $\lambda_\Omega(w_0) = 1$, it certainly need not be the case that the point w_0 is even a local minimum for λ_Ω. However, if we move with steps of hyperbolic size δ along any hyperbolic geodesic starting from w_0, then $\lambda_\Omega(w_0)$ will be the minimum among the values of λ_Ω at these points. If $f \in N_0$ then λ_Ω cannot oscillate between steps of any size δ because of the convexity of λ_Ω along each geodesic.

For functions in N_0 there are several characterizations of John domains in terms of the operator f''/f'. This can be viewed as a strengthening of (2.3)

Theorem 3 *Let $f \in N_0$. The following are equivalent:*

(i) *$f(\mathbf{D})$ is a John domain.*

(ii) $\limsup_{|z|\to 1}(1-|z|^2)\mathrm{Re}\left\{z\frac{f''}{f'}(z)\right\} < 2.$

(iii) $\limsup_{|z|\to 1}(1-|z|^2)\left|\frac{f''}{f'}(z)\right| < 2.$

Proof. $(i) \Rightarrow (ii)$: For convenience, let $\varphi(z) = f''(z)/f'(z)$.

Then by (2.3)

$$|\varphi(z)| \leq \frac{2|z|}{1-|z|^2}\,,$$

and, with $|z| = r$,

$$\begin{aligned} |\varphi'(z)| &= \left|Sf(z) + \frac{1}{2}\varphi(z)^2\right| \\ &\leq \frac{2}{(1-r^2)^2} + \frac{2r^2}{(1-r^2)^2} = \frac{d}{dr}\frac{2r}{1-r^2}\,. \end{aligned} \tag{4.18}$$

Suppose that (ii) does not hold. Then there exists a sequence $z_n \in \mathbf{D}$ with $|z_n| \to 1$ and

$$(1-|z_n|^2)\mathrm{Re}\{z_n\varphi(z_n)\} \to 2. \tag{4.19}$$

Let $x \in (0,1)$ be fixed, set $z_n = \rho_n\zeta_n$, $|\zeta_n| = 1$, and $r_n = (\rho_n - x)/(1 - x\rho_n)$. From (4.18),

$$|\mathrm{Re}\{\zeta_n\varphi(z_n)\} - \mathrm{Re}\{\zeta_n\varphi(r\zeta_n)\}| \leq \int_r^{\rho_n} |\varphi'(t\zeta_n)|\,dt \leq \frac{2\rho_n}{1-\rho_n^2} - \frac{2r}{1-r^2}\,.$$

If $r_n \leq r \leq \rho_n$ then

$$2 - \frac{1-r^2}{r}\mathrm{Re}\{\zeta_n\varphi(r\zeta_n)\} \leq \frac{1-r^2}{1-\rho_n^2}\left(2 - \frac{1-\rho_n^2}{\rho_n^2}\mathrm{Re}\{\zeta_n\varphi(z_n)\}\right).$$

and

$$\frac{1-r^2}{1-\rho_n^2} \leq 2\frac{1-r}{1-\rho_n} \leq 2\frac{1-r_n}{1-\rho_n} = 2\frac{1+x}{1-x}\,.$$

Therefore by (4.19)

$$\left|2 - \frac{1-r^2}{r}\mathrm{Re}\{\zeta_n\varphi(r\zeta_n)\}\right| < \epsilon$$

if $n \geq n_0(\epsilon, x)$.

From these estimates we get

$$\begin{aligned} \log\frac{(1-r_n^2)f'(r_n\zeta_n)}{(1-\rho_n^2)|f'(\rho_n\zeta_n)|} &= \int_{r_n}^{\rho_n}\left(\frac{2r}{1-r^2} - \mathrm{Re}\{\zeta_n\varphi(r\zeta_n)\}\right)dr \\ &< \int_{r_n}^{\rho_n}\frac{\epsilon}{1-r^2}\,dr = \epsilon h_{\mathbf{D}}(r_n\zeta_n, \rho_n\zeta_n) = \epsilon h_{\mathbf{D}}(0,x), \end{aligned}$$

for $n \geq n_0$. Thus

$$\frac{(1-\rho_n^2)|f'(\rho_n\zeta_n)|}{(1-r_n^2)|f'(r_n\zeta_n)|} > e^{-\epsilon h_{\mathbf{D}}(0,x)}.$$

But since $\rho_n = (r_n + x)/(1 + xr_n)$, this contradicts Lemma 2.

$(ii) \Rightarrow (iii)$: If $|\zeta| = 1$ then, by (ii),

$$\begin{aligned} \frac{d}{dr}|\varphi(r\zeta)| &= |\varphi(r\zeta)| \left(\mathrm{Re}\left\{ \zeta \frac{Sf(r\zeta)}{\varphi(r\zeta)} \right\} + \frac{1}{2}\mathrm{Re}\{\zeta\varphi(r\zeta)\} \right) \\ &\leq |Sf(r\zeta)| + \frac{1}{2}|\varphi(r\zeta)|\mathrm{Re}\{\zeta\varphi(r\zeta)\} \\ &< \frac{1+\epsilon}{2(1-r)^2} + \frac{b}{2(1-r)}|\varphi(r\zeta)|, \end{aligned}$$

where $b < 1$, $\epsilon > 0$, and $r_0(\epsilon) < r < 1$. Integrating this linear differential inequality leads to

$$|\varphi(r\zeta)| \leq \frac{1+\epsilon}{(2-b)(1-r)} + O((1-r)^{-b/2}),$$

and, multiplying by $1 - r^2$, this implies (iii) if ϵ is sufficiently small.

$(iii) \Rightarrow (i)$: For $r < \rho < 1$ and $|\zeta| = 1$,

$$\log \frac{(1-\rho^2|)f'(\rho\zeta)|}{(1-r^2)|f'(r\zeta)|} = \int_r^\rho \mathrm{Re}\left\{ \zeta\varphi(t\zeta) - \frac{2t}{1-t^2} \right\} dt.$$

By assumption there is an $0 < \alpha \leq 2$ such that the integral is at most

$$\int_r^\rho \frac{-\alpha}{1-t^2}\, dt = -\alpha h_{\mathbf{D}}(r, \rho).$$

Then

$$\frac{(1-\rho^2)|f'(\rho\zeta)|}{(1-r^2)|f'(r\zeta)|} \leq \left[\left(\frac{1+\rho}{1-\rho} \right) \left(\frac{1-r}{1+r} \right) \right]^{-\alpha/2} \leq 2^{-\alpha/2} \left(\frac{1-\rho}{1-r} \right)^{\alpha/2},$$

which implies that $f(\mathbf{D})$ is a John domain by Corollary 5. This completes the proof.

A related characterization of John domains within the class N_0 is a property of the Poincaré metric. For any simply connected domain Ω one always has the upper bound

$$|\nabla \log \lambda_\Omega| \leq 4\lambda_\Omega. \tag{4.20}$$

This is equivalent to (2.5) and was poined out in [15]. On the other hand, in [3] it was shown that if $f \in N_0^*$ and $\Omega = f(\mathbf{D})$, then there exists a constant $c > 0$ such that

$$|\nabla \log \lambda_\Omega(w)| \geq c|w|\lambda_\Omega(w)^{1/2}. \tag{4.21}$$

The exponent $1/2$ is best possible for the full class N_0^*. Here we can improve this in case the image is a John domain.

Corollary 7 *If $f \in N_0$ then $f(\mathbf{D}) = \Omega$ is a John domain if and only if there exists a constant $c > 0$ such that*

$$|\nabla \log \lambda_\Omega(w)| \geq c|w|\lambda_\Omega(w). \tag{4.22}$$

Proof. Suppose that Ω is a John domain. Then, in particular, Ω is bounded. Once again we consider

$$\eta_\zeta(r) = [(1-r^2)|f'(r\zeta)|]^{-1/2} = \lambda_\Omega^{1/2}(f(r\zeta)),\ |\zeta| = 1,\ 0 \le r < 1,$$

with

$$\frac{\eta_\zeta'}{\eta_\zeta}(r) = \frac{r}{1-r^2} - \frac{1}{2}\mathrm{Re}\left\{\zeta\frac{f''}{f'}(r\zeta)\right\}, \tag{4.23}$$

see (4.16). We also have that

$$\frac{\eta_\zeta'}{\eta_\zeta}(r) = \frac{1}{2}\frac{d}{dr}\log\lambda_\Omega(f(r\zeta)) \le \frac{1}{2}|\nabla\lambda_\Omega(f(r\zeta)||f'(r\zeta)|.$$

By Theorem 3, the right hand side of (4.23) is $\ge c_1 > 0$ if $r_0 \le r < 1$. This proves (4.22) away from $w = 0$, and the estimate at $w = 0$ follows from the fact that, since Ω is bounded, $w = 0$ is the unique critical point of $\log\lambda_\Omega$, see Lemma 2 in [3].

Conversely, suppose that (4.22) holds. Then, with $w = f(z)$, we obtain from (3.5)

$$\begin{aligned}\frac{1}{2}|\nabla\log\lambda_\Omega(w)||f'(z)| &= \left|\frac{\partial}{\partial z}\log\lambda_\Omega(w)\right| = \left|\frac{\bar z}{1-|z|^2} - \frac{1}{2}\frac{f''}{f'}(z)\right| \\ &\ge \frac{1}{2}c|w|\lambda_\Omega(w)|f'(z)|\end{aligned}$$

Therefore

$$\left||z|^2 - \frac{1}{2}(1-|z|^2)z\frac{f''}{f'}(z)\right| \ge \frac{c|z||w|}{2},$$

which implies that

$$\liminf_{|z|\to 1}\left||z|^2 - \frac{1}{2}(1-|z|^2)z\frac{f''}{f'}(z)\right| > 0. \tag{4.24}$$

But now from (2.3) we know that

$$\frac{1}{2}(1-|z|^2)\left|z\frac{f''}{f'}(z)\right| \le |z|^2,$$

which together with (4.24) yields

$$\limsup_{|z|\to 1}\frac{1}{2}(1-|z|^2)\mathrm{Re}\left\{z\frac{f''}{f'}(z)\right\} < 1.$$

This is characterization (ii) of Theorem 3 for John domains.

Theorem 3 and (4.23) show that $f(\mathbf{D})$ is a John domain if and only if there is a constant $c > 0$ such that

$$(1-r^2)\eta_\zeta'(r) \ge c\eta_\zeta(r), \tag{4.25}$$

for $r \geq r_0$. The left hand side of (4.25) must be increasing because of convexity, once again. However, it is possible to construct examples of functions in N_0^* for which $(1-r^2)\eta'_\zeta(r)$ is bounded along some radius $[0, r\zeta)$. In such cases, $f(\mathbf{D})$ cannot be a John domain.

Another interesting example is provided by the normalized extremal $A_t(z)$, (1.4), for the Ahlfors-Weill condition (1.2). In that case $A_t(\mathbf{D})$ is a quasidisk, namely the interior of the intersection of the circles through the points $1/\alpha$, $-1/\alpha$ and $\pm i(1/\alpha)\tan(\pi\alpha/4)$, where $\alpha = \sqrt{1-t}$. Because $|SA_t(z)| \leq SA_t(|z|)$ one can show that the right hand side of

$$(1-r^2)\frac{\eta'_\zeta}{\eta_\zeta}(r) = r - \frac{1}{2}(1-r^2)\mathrm{Re}\left\{\zeta\frac{f''}{f'}(r\zeta)\right\},$$

is smallest when $\zeta = 1$, *i.e.* along the interval $[0,1)$. There one finds that $(1-x^2)\eta'_1/\eta_1(x)$ is increasing near 1 and that it tends to $\sqrt{1-t}$ as $x \to 1$. So the constant c in (4.25) must be less that $\sqrt{1-t}$. We note that the constant must tend to zero as $t \to 1$.

Next, Theorem 3 and its Corollaries gain in significance because of the interesting fact that images of functions in N are quasidisks as soon as they are John domains.

Theorem 4 *Let $f \in N$. If $f(\mathbf{D})$ is a John domain then it is a quasidisk.*

Proof. We shall show that $f(\mathbf{D})$ is linearly connected. Suppose that this is not the case. Then there exist sequences $\zeta_n^\pm$, $|\zeta_n^\pm| = 1$ and z_n on the hyperbolic geodesic between ζ_n^+ and ζ_n^- such that

$$\frac{f(\zeta_n^+) - f(\zeta_m^-)}{f(z_n) - f(\zeta_n^-)} \to 0. \tag{4.26}$$

Let τ_n be the Möbius transformation of the disk onto itself with $\tau_n(0) = z_n$, $\tau_n(\pm 1) = \zeta_n^\pm$ and consider

$$g_n(z) = \frac{f(\tau_n(z)) - f(z_n)}{f'_n(z_n)\tau'_n(0)} = z + b_{2,n}z^2 + \cdots.$$

If

$$h_n(z) = \frac{g_n(z)}{1 + b_{2,n}g_n(z)}$$

then $h_n \in N_0$ and

$$\frac{f(\zeta_n^+) - f(\zeta_m^-)}{f(z_n) - f(\zeta_n^-)} = \frac{h_m(1) - h_m(-1)}{h_m(-1)(b_{2,n}h_n(1) - 1)} \tag{4.27}$$

Since the $|b_{2,n}| \leq 2$ we may assume that $b_{2,n} \to b_2 \in \mathbf{C}$, whence that $h_n \to h$ locally uniformly in $\mathbf{D}$. If the sequence $\{h_n\}$ were bounded then the uniform convergence would be on $\overline{\mathbf{D}}$, and would be to a bounded limit in N_0. Such a limit would be $1:1$ on $\overline{\mathbf{D}}$, and, by (4.26), this would stand in contradiction to the limit in (4.27). Therefore $\{h_n\}$ cannot be bounded, and h must be a rotation of the logarithm L. By rotating f, if necessary, we may assume that $h = L$. Then

$$g_n(z) \to \frac{L(z)}{1 - b_2L(z)}, \quad g'_n(z) \to \frac{1}{(1-z^2)(1-bL(z))^2},$$

locally uniformly in $\mathbf{D}$. Thus

$$\frac{(1-x^2)g_n'(x)}{g_n(x)-g_n(1)} \to \frac{b_2}{1-b_2L(x)}\,, \quad 0<x<1,$$

or, with $\xi_n = \tau_n(x)$,

$$\frac{(1-|\xi_n|^2)|f'(\xi_n)|}{|f(\xi_n)-f(\zeta_n^+)|} \to \frac{|b|}{|1-bL(x)|}\,. \tag{4.28}$$

But the right hand side of (4.28) tends to 0 as $x \to 1$, and this contradicts Corollary 6. The theorem is proved.

Finally, as a counterpoint to the preceding results we are also interested in the question of when a domain $\Omega = f(\mathbf{D})$ *fails* to be a John domain. We find that for a function $f \in N_0$ this will be the case exactly when a sequence of Koebe transforms of f converge to L, so that the sequence of images converges to a parallel strip. Thus through this sequence of Möbius conjugations, which consist of automorphisms of the disk, followed by f, followed by affine transformations, one might say that $\Omega = f(\mathbf{D})$ will not be a John domain if it contains an infinitesimal parallel strip.

The precise statement is as follows.

Theorem 5 *Let $f \in N_0$. Then $f(\mathbf{D}) = \Omega$ is not a John domain if and only if the following two statements hold:*

(i) There exist seqeunces $\{w_n\}$ in Ω and $\{\theta_n\}$ in $\mathbf{R}$ such that for any $\epsilon > 0$,

$$\{w_n + e^{i\theta_n} d(w_n, \partial\Omega)\xi \colon |\mathrm{Re}\,\xi| \le 1/\epsilon\,, |\mathrm{Im}\,\xi| \le (\pi/4) - \epsilon\} \subset \Omega.,$$

if $n \ge n_0(\epsilon)$.

(ii) For each $t \in \mathbf{R}$ there exist points $w_n^\pm \in \partial\Omega$ such that

$$w_n^\pm = w_n + e^{i\theta_n}(t \pm (\pi/4)i)d(w_n, \partial\Omega) + o(d(w_n, \partial\Omega)),$$

as $n \to \infty$.

Proof. Suppose first that Ω is not a John domain. From Lemma 2 and (4.17) we obtain a sequence $\{r_n\}$, $0 \le r_n < 1$ with $r_n \to 1$, and a sequence $\{\zeta_n\}$, $|\zeta_n| = 1$, such that

$$\frac{(1-\rho_n^2)|f'(\rho_n\zeta_n)|}{(1-r_n^2)|f'(r_n\zeta_n)|} \to 1, \quad \rho_n = \frac{r_n + x}{1 + xr_n}\,, \tag{4.29}$$

where $x \in (0,1)$ is fixed. Let $z_n = r_n\zeta_n$, and let

$$g_n(z) = \frac{f\left(\zeta_n \dfrac{z+r_n}{1+r_nz}\right) - f(z_n)}{\zeta_n(1-r_n^2)f'(z_n)} = z + \cdots \tag{4.30}$$

be a sequence of Koebe transforms of f. Then

$$(1-y^2)|g_n'(y)| = \frac{(1-s_n^2)|f'(s_n\zeta_n)|}{(1-r_n^2)|f'(r_n\zeta_n)|}\,, \tag{4.31}$$

for $y \in (0,1)$, where

$$s_n = \frac{r_n + y}{1 + yr_n}\,.$$

At $y = x$ we have

$$(1-x^2)|g_n'(x)| = \frac{(1-\rho_n^2)|f'(\rho_n\zeta_n)|}{(1-r_n^2)|f'(r_n\zeta_n)|} \to 1 \tag{4.32}$$

by (4.29) above.

It follows from (4.15) and (4.16) that $(1-y^2)|g_n'(y)|$ is a decreasing function for $0 < y < 1$. (Actually, $(1-y^2)|g_n'(y)|$ is decreasing for $y \geq -r_n$.)

We may assume that $g_n \to g$ locally uniformly on $\mathbf{D}$. Then $(1-y^2)|g'(y)|$ is non-increasing for $y > 0$. But (4.32) implies that $(1-x^2)|g'(x)| = 1 = g'(0)$. Hence $(1-y^2)|g'(y)| \equiv 1$ for $0 \leq y \leq x$, and then for all $y \in (-1,1)$ because the quantities are analytic. We must therefore have

$$g(z) = \frac{1}{2}\log\frac{1 \pm z}{1 \mp z}\,.$$

Since $g_n \to g$, the Caratheodory Convergence Theorem then implies that

$$g_n(\mathbf{D}) \to g(\mathbf{D}) = \{\xi\colon |\mathrm{Im}\,\xi| < \pi/4\}$$

in the sense of kernel convergence. Therefore, if $\epsilon > 0$ and $n \geq n_0(\epsilon)$, then

$$\{w\colon |\mathrm{Re}\,\xi| \leq 1/\epsilon,\ \ |\mathrm{Im}\,w| \leq (\pi/4) - \epsilon\} \subset g_n(\mathbf{D}). \tag{4.33}$$

Furthermore, if $t \in \mathbf{R}$ then there are $\xi_n^{\pm} \in \partial g_n(\Omega)$ such that

$$\xi_n^{\pm} \to t \pm \frac{\pi}{4}i, \tag{4.34}$$

and

$$d(0,\partial\Omega) \to \frac{\pi}{4}.$$

Let $w_n = f(z_n)$ and $\theta_n = \arg\{\zeta_n f'(z_n)\}$. From (4.30) we can write

$$\Omega = f(\mathbf{D}) = w_n + e^{i\theta_n}(1-|z_n|^2)|f'(z_n)|g_n(\mathbf{D}) \tag{4.35}$$

Then (i) in the statement of the Theorem follows from (4.33)–(4.35), and from (2.12). Next let

$$w_n^{\pm} = w_n + e^{i\theta_n}(1-|z_n|^2)|f'(z_n)|\xi_n^{\pm}.$$

Then $w_n^{\pm} \in \partial\Omega$, and (ii) follows from the above and (2.12), again, since $d(w_n,\partial\Omega) \to 0$.

Conversely, suppose that (i) and (ii) hold. Let $z_n \in \mathbf{D}$ with $f(z_n) = w_n$, and define g_n as in (4.30). The statements (i) and (ii) then imply that $g_n(\mathbf{D}) \to S$, a parallel strip, in the sense of kernel convergence. Moreover, $g'(0) = 1$ forces S to be the strip $|\mathrm{Im}\,\xi| < \pi/4$. Therefore

$$g_n(z) \to \frac{1}{2}\log\frac{1+z}{1-z}\,,$$

and

$$(1-x^2)|g_n'(x)| \to 1, \quad \text{for } 0 \le x < 1.$$

But, working backwards from (4.32), this last statement means that (4.29) holds, and hence that $f(\mathbf{D})$ is not a John domain. This completes the proof.

5 A Comparison Theorem and Well-Accessibility

In the study of criteria for univalence and quasiconformal extension involving the Schwarzian it is often the case that the extremal function satisfies

$$|Sf(z)| \le Sf(|z|). \tag{5.1}$$

The function L is one example of this, as is the extremal function $A_t(z)$, (1.4), for the Ahlfors-Weill criterion, and there are many others. See [14], where a general univalence criterion with this sort of extremal was introduced. Such an inequality leads easily to a comparison theorem for the quotient

$$\frac{|f(s\zeta) - f(r\zeta)|}{(1-r^2)|f'(r\zeta)|}\,, \quad 0 \le r \le s < 1,\ |\zeta| = 1,$$

which we have already encountered in Corollary 6 in connection with well-accessible boundary points and John domains. We shall show for a normalized function that this quotient depends on the size of Sf along a radius $[0,\zeta)$. We state the result in the following form

Lemma 3 *Let f and F be analytic, locally univalent, and normalized functions on $\mathbf{D}$. If $|Sf(z)| \le SF(|z|)$ then for $0 \le r \le s < 1$, $|\zeta| = 1$,*

$$\frac{|f(s\zeta) - f(r\zeta)|}{(1-r^2)|f'(r\zeta)|} \le \frac{F(s) - F(r)}{(1-r^2)F'(r)} \tag{5.2}$$

If equality holds for any pair $r < s$ then $f = F$.

Proof. In Section 2, (2.4) we indicated that a straightforward adaptation of the proof of Lemma 1 in [3] leads to

$$\left|\frac{f''(z)}{f'(z)}\right| \le \frac{F''(|z|)}{F'(|z|)}. \tag{5.3}$$

Furthermore, equality holds for any $z \ne 0$ if and only if $f = F$. Integrating along a radius from $r\zeta$ to $t\zeta$ gives

$$\log\frac{|f'(t\zeta)|}{|f'(r\zeta)|} \le \log\frac{F'(t)}{F'(r)},$$

Integrating again with respect to t from $t = r$ to $t = s$, we obtain

$$\frac{|f(s\zeta) - f(r\zeta)|}{|f'(r\zeta)|} \leq \frac{F(s) - F(r)}{F'(r)},$$

which is (5.2). The case of equality in (5.2) follows from that in (5.3).

Incidentally, it was only for the case of equality that we used analyticity of F. We could have stated the Lemma, minus that part, for a smooth, real-valued, normalized function F on $[0, 1)$.

From the Lemma we can now deduce a version of Theorem 4 that uses the local information of well-accessibility.

Theorem 6 *Let $f \in N_0$ and suppose that $|Sf(z)| \leq Sf(|z|)$. The following are equivalent.*
(i) $f(\mathbf{D})$ is a John domain.
(ii) $f(\mathbf{D})$ is a quasidisk.
(iii) $f(\zeta)$ is well-accessible for all $|\zeta| = 1$.
(iv) $f(1)$ is well-accessible.

Proof. Theorem 4 gives $(i) \Rightarrow (ii)$, and the implications $(ii) \Rightarrow (iii) \Rightarrow (iv)$ are clear. It is only $(iv) \Rightarrow (i)$ that requires some proof. Suppose $f(1)$ is well-accessible. According to the proof of Corollary 6,

$$\sup_{r<1} \frac{|f(1) - f(r)|}{(1 - r^2)|f'(r)} \leq M < \infty,$$

Lemma 3 implies that for any $|\zeta| = 1$

$$\frac{|f(s\zeta) - f(r\zeta)|}{(1 - r^2)|f'(r\zeta)|} \leq \frac{f(s) - f(r)}{(1 - r^2)f'(r)},$$

for $0 \leq r \leq s < 1$. Thus

$$\sup_{|\zeta|=1} \sup_{0 \leq r \leq s < 1} \frac{|f(s\zeta) - f(r\zeta)|}{(1 - r^2)|f'(r\zeta)|} \leq \sup_{r<1} \frac{f(1) - f(r)}{(1 - r^2)|f'(r)} \leq M,$$

and by Corollary 6 again we conclude that $f(\mathbf{D})$ is a John domain.

We can also obtain a negative statement on well-accessibility. Let $f \in N_0$ and let u be the solution of

$$u'' + \frac{1}{2}(Sf)u = 0, \quad u(0) = 1, \; u'(0) = 0, \tag{5.4}$$

so that

$$f(z) = \int_0^z u^{-2}(\zeta)\, d\zeta.$$

The function $|u|$ is never zero, and when restricted to a radius $[0,\zeta)$, $|\zeta|=1$ it satisfies

$$|u|'' + \frac{1}{2}\sigma_\zeta(r)|u| = 0,$$

where $'$ denotes differentiation with respect to r and

$$\sigma_\zeta(r) = \mathrm{Re}\{\zeta^2 Sf(r\zeta)\} - \frac{1}{2}\left[\mathrm{Im}\left\{\zeta\frac{f''}{f'}(r\zeta)\right\}\right]^2. \tag{5.5}$$

Theorem 7 *If $f \in N_0$ and $\liminf_{r\to 1}(1-r^2)^2\sigma_\zeta(r) \geq 2$, then $f(\zeta)$ is not well accessible.*

The hypothesis that $f \in N_0$ is actually more than we need. All that is necessary is that the solution u of (5.4) never vanish.

Proof. Suppose by way of contradiction that $f(\zeta)$ is well-accessible. Then, referring back to (4.11) and to the discussion following Corollary 5 in the previous Section, there is a constant $M > 0$, depending on ζ and a constant $\alpha > 0$, such that

$$|f'(s\zeta)| \leq M|f'(r\zeta)|\left(\frac{1-s}{1-r}\right)^{\alpha-1} \tag{5.6}$$

for $0 \leq r \leq s < 1$. Define the positive function

$$\phi(x) = \int_0^x |f'(t\zeta)|\,dt$$

on $[0,1)$. Then ϕ is normalized and $S\phi(x) = 2\sigma_\zeta(x)$. From (5.6) it follows that

$$\phi(1) - \phi(x) \leq \frac{M|f'(x\zeta)|}{(1-x)^{\alpha-1}}\int_x^1 (1-t)^{\alpha-1}\,dt = \frac{M}{\alpha}|F'(x\zeta)|(1-x).$$

Hence

$$\frac{\phi(1)-\phi(x)}{(1-x^2)\phi'(x)} \leq \frac{M}{\alpha} = M_1. \tag{5.7}$$

We claim that (5.7) implies the existence of an $\epsilon > 0$ such that the solution v of

$$v'' + \frac{1}{2}\left(\sigma_\zeta(x) + \frac{2\epsilon}{(1-x^2)^2}\right)v = 0, \quad v(0)=1,\ v'(0)=0 \tag{5.8}$$

is positive. This will result in a contradiction, because

$$\sigma_\zeta(x) + \frac{2\epsilon}{(1-x^2)^2} \geq \frac{2+\epsilon}{(1-x^2)^2},$$

if $x \geq x_0(\epsilon)$, and therefore any solution of (5.8) must be oscillatory by [8], Chapter XI, Theorem 7.1 (disguised).

To show this, let $a = \phi(1)$ and consider on $[0, a)$ the 'Poincaré metric'

$$\lambda(y) = \frac{1}{(1-x^2)\phi'(x)}\,, \quad y = \phi(x).$$

We claim that for $\epsilon > 0$ small enough the solution w of

$$w'' + \frac{1}{2}\epsilon\lambda^2(y)w = 0, \quad w(0) = 1,\; w'(0) = 0 \tag{5.9}$$

is positive. For this, by (5.7)

$$\epsilon\lambda(y)^2 = \frac{\epsilon}{(1-x^2)^2\phi'(x)^2} \leq \frac{M_1^2\epsilon}{(a-y)^2} \leq \frac{4a^2M_1^2\epsilon}{(a^2-y^2)^2}\,, \tag{5.10}$$

and the solution of

$$\omega'' + \frac{4a^2M_1^2\epsilon}{(a^2-y^2)^2}\omega = 0, \quad \omega(0) = 0,\; \omega'(0) = 1$$

is positive as soon as $4a^2M_1^2\epsilon \leq 1$; see [9], p. 492 for the explicit solution. Thus by (5.10) and the standard Sturm comparison theorem, the solution w of (5.9) will be positive for $\epsilon \leq 1/2a^2M_1^2$.

With such a solution w we define

$$v(x) = |u(x\zeta)|w(\phi(x)), \quad u = (f')^{-1/2}.$$

A straightforward computation shows that v is the desired positive solution of (5.8). The contradiction obtains, and the Theorem is proved.

The results in this Section are quite useful in analyzing examples.

6 Two Examples

Once again we recall that a quasidisk is a linearly connected John domain, and that, within N, if the image of the disk is a John domain then it is linearly connected and hence a quasidisk. In this Section we present two examples. One to show that the image of a function in N may be linearly connected without being a John domain, and one to show that the image need not be linearly connected. Möbius conjugations of L can be used to provide trivial examples, thus the burden is to give examples in N_0^*

Theorem 8 *There exists a function $f \in N_0^*$ such that $f(\mathbf{D})$ is linearly connected, but not a John domain.*

Proof. The example is the normalized solution f of

$$Sf(z) = \frac{1+z^2}{(1-z^2)^2}\,. \tag{6.1}$$

First, we see immediately that $f \in N_0^*$, and since f is not a rotation of L it must be bounded. Notice also that $|Sf(z)| \le Sf(|z|)$, the type of function we considered in the previous Section. Thus, by Theorem 6, to see that $f(\mathbf{D}) = \Omega$ is not a John domain it suffices to check that $f(1)$ is not well-accessible. But from (5.5), $\sigma_1(x) = Sf(x)$, and $\lim_{x\to 1}(1-x^2)^2\sigma_1(x) = 2$. The desired conclusion now follows from Theorem 7.

Next, because Sf is even, f is odd, and because f is real on the real axis Ω is symmetric with respect to both the real and imaginary axes. Finally, the Schwarzian is analyitic on $\overline{\Omega}\backslash\{1,1\}$, and it follows that $\partial\Omega$ is an analytic curve away from $\pm f(1)$. Hence to complete the proof it suffices to show that Ω is linearly connected in a neighborhood of $f(1)$.

We have the representation

$$f(z) = \int_0^z u^{-2}(\zeta)\, d\zeta.$$

where u is the solution of the initial value problem

$$u'' + \frac{1}{2}(Sf)u = 0, \quad u(0) = 1,\ u'(0) = 0. \tag{6.2}$$

Since $Sf(x) \ge 0$ for $0 \le x < 1$, the function u is decreasing. We must have $u(1) = 0$, for if not, then, as

$$|f'(z)| \le f'(|z|) \le u^{-2}(1) < \infty,$$

we would conclude that f is Lipschitz. This is not the case, because the John condition is violated near $\pm f(1)$.

The possible orders of vanishing of u come from an analysis of the solutions of (6.2) at 1. Since $\lim_{x\to 1}(1-x^2)^2 Sf(x) = 1$, the indicial equation is $(\rho - (1/2))^2 = 0$. Therefore, as $x \to 1$,

$$u(x) \sim (1-x)^{1/2}.$$

or

$$u(x) \sim (1-x)^{1/2}(1 + a\log(1-x)),\ a \ne 0.$$

The first case would give $f(1) = \infty$, and so the second must obtain. From this explicit information, it is not difficult to see that a neighborhood of $f(1)$ is the Lipschitz image of a neighborhood of $g(1)$, where

$$g(z) = c_1 + \frac{c_2}{1 + a\log(1-z)},$$

Thus near $f(1)$ the domain Ω is linearly connected.

The construction of our second example is much more involved.

Theorem 9 *There exists a function $f \in N_0^*$ such that $f(\mathbf{D})$ is not linearly connected.*

Proof. Let $0 < \delta_n < 1$ be a sequence decreasing to zero with, at least,

$$\delta_{n+1} \le \frac{1}{4}\delta_n, \quad n = 0, 1, \ldots \tag{6.3}$$

Faster rates of decay will be required as the construction proceeds. Next let

$$x_n = 1 - \delta_n, \quad \tau_n(z) = \frac{x_n - z}{1 - x_n z}, \tag{6.4}$$

$$\varphi_n(z) = \frac{\tau_n{}'(z)}{1 + \tau_n(z)^2} = -\frac{1 - x_n^2}{(1 - x_n z)^2 + (x_n - z)^2}. \tag{6.5}$$

Finally, put

$$a_n = 2 - \frac{1}{2^n}, \tag{6.6}$$

and define

$$\psi(z) = \sum_{n=1}^{\infty} a_n \varphi_n(z)^2. \tag{6.7}$$

The function ψ is analytic on the disk.

We let f be the normalized solution to

$$Sf = \psi.$$

The problem is to choose the sequence $\{\delta_n\}$ so that $f \in N_0^*$ and $f(\mathbf{D})$ is not linearly connected.

We show first that with appropriate choices of the δ_n we can obtain

$$(1 - |z|^2)^2 |\psi(z)| < 2. \tag{6.8}$$

To begin with,

$$(1 - |z|^2)|\varphi_n(z)| = \frac{1 - |\tau_n(z)|^2}{|1 + \tau_n(z)^2|} \leq 1, \tag{6.9}$$

while (6.4) and (6.5) imply also that

$$\begin{aligned} (1 - |z|^2)|\varphi_n(z)| &\leq \frac{(1 - |z|^2)(1 - x_n^2)}{|(1 - x_n z)^2 + (x_n - z)^2|} \\ &\leq \frac{4\delta_n |1 - z|}{|(1 + x_n^2)(1 + z^2) - 4x_n z|}. \end{aligned} \tag{6.10}$$

Now let

$$H_n = \{z \in \mathbf{D} \,:\, (\delta_n \delta_{n+1})^{1/2} \leq |z - 1| < (\delta_{n-1}\delta_n)^{1/2}\,\}, \quad n = 1, 2, \ldots,$$

$$H_0 = \{z \in \mathbf{D} \,:\, (\delta_0 \delta_1)^{1/2} \leq |z - 1|\,\},$$

so that

$$\mathbf{D} = \bigcup_{n=0}^{\infty} H_n\,.$$

To estimate $(1-|z|^2)^2|\psi(z)|$ on $\mathbf{D}$ we estimate $(1-|z|^2)|\varphi_n(z)|$ on H_m.

For the term in (6.7) with $n=m$ we simply use (6.9). For the terms with $n \le m-1$ we estimate the denominator in (6.10) by writing

$$\begin{aligned} |(1+x_n^2)(1+z^2)-4x_nz| &= |2\delta_n^2 - 2\delta_n^2(1-z)+(1+x_n^2)(1-z)^2| \\ &\ge 2(\delta_n^2 - \delta_n|1-z|-|1-z|^2). \end{aligned}$$

If $z \in H_m$, $m \ge 1$, then using $|z-1| \le (\delta_{m-1}\delta_m)^{1/2}$ gives

$$(1-|z|^2)|\varphi_n(z)| \le \frac{2\delta_n(\delta_{m-1}\delta_m)^{1/2}}{\delta_n^2 - \delta_n(\delta_{m-1}\delta_m)^{1/2} - \delta_{m-1}\delta_m}.$$

Since $\delta_m \le (1/4)\delta_{m-1} \le (1/4)\delta_m$ and $\delta_{m-1} \le 4^{n-m+1}\delta_n$, we can then conclude that

$$(1-|z|^2)|\varphi_n(z)| \le \frac{2\delta_n(\delta_{m-1}\delta_m)^{1/2}}{\frac{1}{4}\delta_n^2} \le 8\cdot 4^{n-m+1}\left(\frac{\delta_m}{\delta_{m-1}}\right)^{1/2}. \qquad (6.11)$$

Next take $n \ge m+1$. This time we write

$$\begin{aligned} |(1+x_n^2)(1+z^2)-4x_nz| &= |2(1-z)^2+2\delta_n^2 z-(2\delta_n-\delta_n^2)(1-z)^2| \\ &\ge 2((1-2\delta_n)|1-z|^2-\delta_n^2). \end{aligned}$$

If $z \in H_m$, $m \ge 0$ then $(\delta_m\delta_{m+1})^{1/2} \le |1-z|$ and

$$\begin{aligned} (1-|z|^2)|\varphi_n(z)| &\le \frac{2\delta_n|1-z|}{(1-2\delta_n)\delta_m\delta_{m+1}-\delta_m^2} \le \frac{2\delta_n(\delta_m\delta_{m+1})^{1/2}}{(1-2\delta_n)\delta_m\delta_{m+1}-\delta_n^2} \\ &\le \frac{2\delta_n(\delta_m\delta_{m+1})^{1/2}}{(\frac{1}{2}-\frac{1}{4})\delta_m\delta_{m+1}} \\ &= 8\frac{\delta_n}{\delta_{m+1}}\left(\frac{\delta_{m+1}}{\delta_m}\right)^{1/2}, \end{aligned}$$

where in the penultimate line we used $\delta_n^2 \le (1/4)\delta_m\delta_{m+1}$ and $\delta_n \le 1/4$. With $\delta_n \le 4^{-n+m-1}\delta_{m+1}$, we have the estimate

$$(1-|z|^2)|\varphi_n(z)| \le 8\cdot 4^{-n+m-1}\left(\frac{\delta_{m+1}}{\delta_m}\right)^{1/2}. \qquad (6.12)$$

From the definition of ψ in (6.7) together with the estimates (6.9), (6.11) and (6.12) we have for $z \in H_m$, $m \ge 0$, that

$$\begin{aligned} (1-|z|^2)|\psi(z)| &\le 2-\frac{1}{2^m}+C_1\sum_{n=1}^{m-1}4^{2(n-m)}\frac{\delta_m}{\delta_{m-1}}+C_2\sum_{n=m+1}^{\infty}4^{2(m-n)}\delta_{m+1}\delta_m \\ &\le 2-\frac{1}{2^m}+C_3\max\left\{\frac{\delta_m}{\delta_{m-1}},\frac{\delta_{m+1}}{\delta_m}\right\}. \qquad (6.13) \end{aligned}$$

In this and subsequent estimates we let $C_1, C_2, \ldots$ denote absolute constants. This last expression will be < 2 if $\delta_m \to 0$ sufficiently rapidly.

Let h_m be the normalization of $f \circ \tau_m$, i.e., $h_m(0) = 0$, $h_m'(0) = 1$, $h_m''(0) = 0$. For $y \in (-1, 1)$ it is easy to check that $\tau_m(iy) \in H_m$ for $m \geq 1$. We then have

$$\begin{aligned}(1-y^2)^2 Sh_m(iy) &= (1-y^2)^2 \tau_m'(iy)^2 \psi(\tau_m(iy)) \\ &= 2 - \frac{1}{2^m} + \sum_{n \neq m} a_n (1-y^2)^2 \tau_m'(iy)^2 \varphi_n(\tau_m(iy))^2.\end{aligned}$$

By virtue of the invariance of the hyperbolic metric, the sum is in absolute value less than

$$\sum_{n \neq m} (1-|z|^2)^2 |\varphi_n(z)|^2, \quad z = \tau_m(iy) \in H_m\,,$$

As in the estimate (6.13) above, the sum will tend to 0 if we now require that $\delta_{m+1}/\delta_m \to 0$. Hence

$$(1-y^2)^2 Sh_m(iy) \to 2 \tag{6.14}$$

as $m \to \infty$, uniformly for y in any compact subset of $(-1, 1)$.

We may assume that $h_m \to h$ locally uniformly in $\mathbf{D}$. Then from (6.14)

$$(1-y^2)^2 Sh(iy) = 2, \quad y \in (-1, 1),$$

and consequently

$$h(z) = \frac{i}{2} \log \frac{1-iz}{1+iz}\,.$$

This in turn implies that

$$h_m(iy) \to \frac{i}{2} \log \frac{1+y}{1-y}\,, \tag{6.15}$$

uniformly in y.

Let

$$g_m(z) = \frac{f(\tau_m(z)) - f(x_m)}{(1-x_m^2) f'(x_m)}\,, \tag{6.16}$$

so that

$$h_m = \frac{g_m}{1 + b_{2,m} g_m}\,,$$

where

$$b_{2,m} = \frac{1}{2}(1-x_m^2)\frac{f''}{f'}(x_m) - x_m,$$

see (2.13). Note that $g_m(0) = 0$ and $g_m'(0) = 1$. We may assume that $b_{2,m} \to b_2$, and so by (6.15),

$$g_m(iy) \to \frac{\frac{i}{2} \log \frac{1+y}{1-y}}{1 - \frac{ib_2}{2} \log \frac{1+y}{1-y}}\,. \tag{6.17}$$

Our final choice of the $\{\delta_n\}$ will guarantee that $b_2 \neq 0$. This is essential for the last step in the proof, and we show now how to make the estimate needed to deduce it.

First note the general inequality

$$(1-x^2)\left|\frac{f''}{f'}(x)\right| \leq (1-x^2)^{1/2}\int_0^x (1-t^2)^{1/2}|Sf(t)|\,dt\,,$$

for $0 \leq x < 1$, when $f \in N_0$. This follows from

$$\begin{aligned}
\frac{d}{dx}\left((1-x^2)^{1/2}\left|\frac{f''}{f'}(x)\right|\right) &\leq -\frac{x}{(1-x^2)^{1/2}}\left|\frac{f''}{f'}(x)\right| + (1-x^2)^{1/2}\left|\left(\frac{f''}{f'}\right)'(x)\right| \\
&\leq -\frac{x}{(1-x^2)^{1/2}}\left|\frac{f''}{f'}(x)\right| + (1-x^2)^{1/2}|Sf(x)| \\
&\qquad + \frac{(1-x^2)^{1/2}}{2}\left|\left(\frac{f''}{f'}\right)^2(x)\right| \\
&\leq (1-x^2)^{1/2}|Sf(x)|,
\end{aligned}$$

where we have used

$$\left|\frac{f''}{f'}(x)\right| \leq \frac{2x}{1-x^2}\,,$$

from (2.3).

For our normalized function f with $Sf = \psi$, we obtain from this and from (6.3)–(6.7)

$$\begin{aligned}
(1-x_m)^2\left|\frac{f''}{f'}(x_m)\right| &\leq 4\delta_m^{1/2}\int_0^{x_m}(1-t)^{1/2}\sum_{n=1}^{\infty}\frac{\tau_n'(t)^2}{(1+\tau_n(t))^2}\,dt \\
&\leq 4\delta_m^{1/2}\int_0^{x_m}\sum_{n=1}^{\infty}\frac{(1-t)^{1/2}(1-x_n)^2}{(1-x_nt)^4}\,dt \\
&\leq 4\delta_m^{1/2}\int_0^{x_m}\sum_{n=1}^{\infty}\frac{(1-x_n)^2}{(1-x_nt)^{7/2}}\,dt \\
&\leq \frac{8}{5}\,\delta_m^{1/2}\sum_{n=1}^{\infty}\frac{(1-x_n^2)^2}{x_n(1-x_nx_m)^{5/2}} \\
&\leq \frac{8}{5}\,\delta_m^{1/2}\left\{\frac{1}{x_m(1-x_m^2)^{1/2}} + C_4\sum_{n=1}^{m-1}\frac{1}{(1-x_n)^{1/2}}\right. \\
&\qquad \left. + C_5\sum_{n=m+1}^{\infty}\frac{(1-x_n)^2}{(1-x_m)^{5/2}}\right\} \\
&\leq \frac{8}{5}\frac{1}{x_m} + C_6\sum_{n=1}^{m-1}\left(\frac{\delta_m}{\delta_n}\right)^{1/2} + C_7\sum_{n=m+1}^{\infty}\left(\frac{\delta_n}{\delta_m}\right)^2 \\
&\leq \frac{8}{5}\frac{1}{x_m} + C_6\left(\frac{\delta_m}{\delta_{m-1}}\right)^{1/2}\sum_{n=1}^{m-1}2^{n-m+1} \\
&\qquad + C_7\left(\frac{\delta_{m+1}}{\delta_m}\right)^2\sum_{n=m+1}^{\infty}16^{m-n+1}\,.
\end{aligned}$$

If δ_m decreases to zero rapidly enough, then for large m we can make sure that

$$(1-x_m^2)\left|\frac{f''}{f'}(x_m)\right| \leq c < 2.$$

Returning to the previous considerations, it then follows that

$$b_{2,m} = \frac{1}{2}(1-x_m^2)\frac{f''}{f'}(x_m) - x_m \to b_2 \neq 0,$$

since $x_m \to 1$.

Let $f(\mathbf{D}) = \Omega$ and $g_m(\mathbf{D}) = \Omega_m$. According to Proposition 5.6 in [18], if Ω is linearly connected then f extends continuously to $\overline{\mathbf{D}}$, and there is a constant A such that

$$\operatorname{diam} f(\gamma) \leq A|f(z_1) - f(z_2)|$$

for $z_1,\ z_2 \in \overline{\mathbf{D}}$, where γ is the hyperbolic geodesic between z_1 and z_2. Suppose that this is so. Clearly the constant A is an affine invariant of the domain, and hence by (6.16) the Ω_m are also all linearly connected with the same constant A. Let γ be the geodesic from $-iy$ to iy, with $0 < y < 1$. On the one hand, $0 = g_m(0) \in g_m(\gamma)$, and $g_m'(0) = 1$, so by the Koebe 1/4-Theorem the diameters of the $g_m(\gamma)$ are uniformly bounded away from zero if $y_0 \leq y < 1$. On the other hand, by (6.17) and the fact that $b_2 \neq 0$, we conclude that $|g_m(iy) - g_m(-iy)|$ can be made arbitrarily small for m sufficiently large and y close to 1. This contradiction shows that the $\Omega = f(\mathbf{D})$ is not linearly connected.

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